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Thermal characteristics of slug flow in rectangular ducts

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Abstract—A rigorous solution is obtained for the temperature field and the Nusselt numbers in the fully developed thermal region of rectangular ducts, in which a uniform velocity profile occurs (slug flow). The thermal boundary conditions T (constant wall temperature peripherally as well as axially), H1 (constant axial wall heat flux with a constant peripheral wall temperature) and H2 (constant axial wall heat flux with uniform peripheral wall heat flux) are examined. The 2-D temperature distribution and the Nusselt numbers are calculated as functions of the aspect ratio. The results, in terms of temperature profiles and Nusselt numbers, are presented and discussed in tables and graphs, considering all the possible combinations of heated and adiabatic walls of the rectangular cross section. © Elsevier, Paris.

fully developed thermal region / Nusselt number / slug flow / T, H1 and H2 boundary conditions / rectangular ducts

Résumé — Analyse thermique des conduits à section rectangulaire dans le cas d'un écoulement à distribution uniforme des vitesses. Cet article présente une méthode analytique pour déterminer la distribution 2D de la température et les nombres de Nusselt dans la région thermique entièrement développée des conduits à section rectangulaire, où un profil uniforme de vitesse se produit. Les conditions aux limites de type T (température constante à la paroi aussi bien qu'axialement), H1 (flux de chaleur axialement constant et température périphérique constante) et H2 (flux de chaleur axialement constant et flux de chaleur à la paroi uniforme) ont été examinés. La distribution 2D de la temperature et les nombres de Nusselt sont calculés en fonction du rapport entre les côtés de la section rectangulaire. Les données sont présentées dans cet article sous forme de tableaux et figures, et considèrent toutes les combinaisons possibles de parois adiabatiques et chauffées qui peuvent caractériser un conduit rectangulaire. © Elsevier, Paris.

région thermique entièrement développée / nombre de Nusselt / écoulement à distribution uniforme des vitesses / conditions aux limites de type T, H1 et H2 / conduit rectangulaire

Nomenclature

a,b	longer and shorter sides, respectively, of the rectangular cross section	m
$B_{n,m}$	coefficients defined in equation (15)	
c^*	function of the combination of heated and adiabatic walls	
$C_{n,m}$	coefficients defined in equation (16)	
c_p	fluid specific heat	$J{\cdot}kg^{-1}{\cdot}K^{-1}$
D_h	hydraulic diameter of the duct $2\,a\,b/(a+b)$	m
h	heat transfer coefficient	$W{\cdot}m^{-2}{\cdot}K^{-1}$
m,n	summation integer indices	
K	fluid thermal conductivity	$W{\cdot}m^{-1}{\cdot}K^{-1}$
L_h	heated perimeter length	m
L_h^*	dimensionless heated perimeter length L_h/a	
Nu	Nusselt number $h\,D_h/K$	
q'	thermal power per unit of length	$W{\cdot}m^{-1}$
q''	thermal power per unit of area (heat flux)	$W{\cdot}m^{-2}$
$T(.)$	dimensionless fluid temperature	
V	fluid velocity	$m{\cdot}s^{-1}$
x,y,z	dimensionless rectangular Cartesian co-ordinates	

Greek symbols

α	fluid thermal diffusivity	$m^2{\cdot}s^{-1}$
β	aspect ratio $b/a < 1$	
ρ	fluid density	$kg{\cdot}m^{-3}$
$\theta(.)$	fluid temperature	K
ξ,η,ζ	Cartesian co-ordinates	m

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Subscripts

b bulk quantity
w quantity evaluated at the wall
0 inlet quantity

1. INTRODUCTION

Analysis of the heat transfer behaviour of newtonian fluids in rectangular ducts is a topic of special interest in compact heat exchangers, such as radiators or condensers in air-conditioning units [1, 2]. The theoretical analysis for rectangular geometry is more complex and rarer than in the case of circular pipe flow. In fact, the investigation of rectangular ducts is very complicated because it requires a two-dimensional analysis. Generally, the thermal boundary conditions are also complex because there are many possible ways of imposing different temperatures or heat fluxes on the four wetted sides. Hence a clear understanding of the thermal boundary conditions is essential. Due to their practical application, the most interesting boundary conditions are the well-known T, H1 and H2 conditions as defined by Shah and London [3].

For the T boundary condition, the wall temperature is considered to be constant both peripherally and axially. This situation occurs in many practical applications such as condensers, evaporators, automotive radiators (having high liquid flow rates), with negligible wall thermal resistance.

For non-circular ducts with corners or variable peripheral curvature heated, for example, with an electric resistance, all with negligible normal wall thermal resistance, two cases may occur:

a) for highly conductive materials (e.g., copper, aluminium) the axial wall heat flux may be considered to be constant with uniform peripheral wall temperature (H1 boundary condition);

b) for very low conductive materials (e.g., glass-ceramic, teflon) with the duct having uniform wall thickness, the axial wall heat flux can be fixed as constant with a uniform peripheral wall heat flux (H2 boundary condition).

Moreover, many different situations can be considered, assuming a particular condition for every side of the rectangle, as described in Section 2.2 of the present paper: in the literature, eight classic thermal versions are proposed for the T, H1 and H2 problems.

For any boundary condition, extensive numerical, analytical and experimental studies have been carried out both for laminar fully developed flow [4, 5] and for slug flow [6], as well as for the thermal entrance region and for the thermal fully developed region in rectangular ducts [7]. In particular, many authors have studied the heat transfer behaviour for the slug flow [8, 9, 10, 11, 12], because this flow condition can be regarded as a simplified model for the analysis of turbulent flow, of laminar flow in the hydrodynamic entrance region of a fluid with negligible Prandlt number, and of fully developed laminar flow of a pseudoplastic fluid with a vanishing power-law index. Moreover, slug flow forced convection can describe a solid body moving, with good thermal contact, through a heated sleeve.

The aim of this paper is the rigorous determination of the temperature profile and the Nusselt numbers of a fluid with uniform velocity profile through rectangular ducts for the T, H1 and H2 boundary conditions. These results enrich and complete the previous analysis in this field.

2. BASIC EQUATIONS

2.1. Energy equation

Consider a steady laminar slug flow in the thermally developed region of a rectangular duct with axially unchanging cross-section. A Cartesian system of co-ordinates ξ, η, ζ is assumed, with its origin in the left bottom corner of the inlet rectangular cross section (η along the short side b, ζ perpendicular to the cross section). The fluid has a uniform velocity V and an inlet temperature $\theta_0(\zeta = 0)$. Under the assumption of constant fluid properties and neglecting axial thermal conduction, natural convection, viscous dissipation and internal energy sources, with rigid and non-porous duct walls, the differential steady state energy equation may be written as:

$$\frac{\partial^2\theta}{\partial\xi^2} + \frac{\partial^2\theta}{\partial\eta^2} = \frac{V}{\alpha}\frac{\partial\theta}{\partial\zeta} \tag{1}$$

In order to solve equation (1) an energy balance between section ζ and $\zeta - d\zeta$ gives the axial variation of the fluid bulk temperature θ_b for the boundary conditions T, H1 and H2:

$$\begin{cases} \dfrac{\partial\theta_b}{\partial\zeta} = \dfrac{h\,L_h(\theta_w - \theta_b)}{\rho\,c_p\,V\,a\,b} \ (\mathrm{T}); \\[2ex] \dfrac{\partial\theta_b}{\partial\zeta} = \dfrac{q'}{\rho\,c_p\,V\,a\,b} \ (\mathrm{H1}); \\[2ex] \dfrac{\partial\theta_b}{\partial\zeta} = \dfrac{q''\,L_h}{\rho\,c_p\,V\,a\,b} \ (\mathrm{H2}) \end{cases} \tag{2}$$

where L_h is the length of the heated rectangular perimeter, θ_w is the wall temperature and q' and q'' are the thermal power per unit of length and per unit of area respectively, imposed on the heated walls of the rectangular duct.

In the fully developed thermal region of a heated duct the temperature profile continues to change with ζ but the 'relative shape' G of the profile no longer changes:

$$\frac{\partial}{\partial\zeta}\left[\frac{\theta_w - \theta}{\theta_w - \theta_b}\right] = \frac{\partial G}{\partial\zeta} = 0 \tag{3}$$

Using equation (3) it is possible to demonstrate that, in the thermal fully developed region, the following ensues:

$$\frac{\partial \theta_b}{\partial \varsigma} = \frac{1}{G}\frac{\partial \theta}{\partial \varsigma}\ (\mathrm{T});\ \frac{\partial \theta_b}{\partial \varsigma} = \frac{\partial \theta}{\partial \varsigma}\ (\mathrm{H1\ and\ H2}) \qquad (4)$$

It is appropriate to introduce the dimensionless coordinates:

$$x = \frac{\xi}{a}\quad 0 \le x \le 1;\quad y = \frac{\eta}{a}\quad 0 \le y \le \beta = \frac{b}{a}; \qquad (5)$$

and using different dimensionless temperatures for the three boundary conditions examined:

$$\begin{aligned} T &= \frac{\theta_w - \theta}{\theta_w - \theta_0}\ (\mathrm{T}); \\ T &= \frac{K(\theta - \theta_0)}{q'}\ (\mathrm{H1}); \\ T &= \frac{K(\theta - \theta_0)}{q'' D_h}\ (\mathrm{H2}) \end{aligned} \qquad (6)$$

Consequently, the dimensionless energy balance equation is readily obtained in the following forms for the T, H1 and H2 problems examined:

$$\begin{cases} \left(\dfrac{\partial^2 T}{\partial x^2} + \dfrac{\partial^2 T}{\partial y^2}\right) = -c^*\, Nu\, T(x,y)\ (\mathrm{T}) \\ \left(\dfrac{\partial^2 T}{\partial x^2} + \dfrac{\partial^2 T}{\partial y^2}\right) = \dfrac{1}{\beta}\ (\mathrm{H1}) \\ \left(\dfrac{\partial^2 T}{\partial x^2} + \dfrac{\partial^2 T}{\partial y^2}\right) = c^*\ (\mathrm{H2}) \end{cases} \qquad (7)$$

where c^* is $L_h^*(1+\beta)/(2\beta^2)$ and L_h^* is L_h/a.

2.2. Thermal versions

In order to solve equation (7), one can remark that c^* depends on the heated length of the rectangular perimeter, and hence the energy equation for the T and H2 problems changes if different temperatures or heat fluxes are imposed on the four wetted sides of the rectangular duct. In this paper the eight thermal versions which in the literature have been proposed for the T, H1 and H2 problems are considered; the following nomenclature is usually assumed in the analysis of rectangular ducts for the eight thermal versions considered:

4 – four constant wall temperature (T, H1) or heated (H2) sides;

3L – three constant wall temperature (T, H1) or heated (H2) sides and one adiabatic short side;

3S – three constant wall temperature (T, H1) or heated (H2) sides and one adiabatic long side;

2L – two constant wall temperature (T, H1) or heated (H2) sides and two adiabatic short sides;

2S – two constant wall temperature (T, H1) or heated (H2) sides and two adiabatic long sides;

2C – one short and one long constant wall temperature (T, H1) or heated (H2) sides (corner version);

1L – one constant wall temperature (T, H1) or heated (H2) long side;

1S – one constant wall temperature (T, H1) or heated (H2) short side.

2.3. The bulk temperature and the Nusselt number

The knowledge of temperature distribution over a cross section of the rectangular duct allows the determination of the bulk temperature:

$$T_b = \frac{1}{\beta}\int_0^1 \int_0^\beta T(x,y)\,\mathrm{d}x\,\mathrm{d}y \quad (\text{for T, H1 and H2}) \qquad (8)$$

Finally, the Nusselt number can be obtained by an energy balance on the heated perimeter of the rectangular duct; its expression is:

$$\begin{aligned} Nu_T &= -\frac{1+\beta}{2\beta T_b}\,\frac{\left(\displaystyle\int_{L_h^*} \left.\frac{\partial T}{\partial n}\right|_P \mathrm{d}l\right)}{L_h^*}; \\ Nu_{H1} &= \frac{1}{c^*\beta\,(T_w - T_b)}; \\ Nu_{H2} &= \frac{1}{\left(\displaystyle\int_{L_h^*} T|_P\,\mathrm{d}l\right) - T_b} \end{aligned} \qquad (9)$$

where P is a generic point in the duct heated perimeter (L_h^*), n is the normal direction of the heated wall and $\mathrm{d}l$ is an infinitesimal element of the heated perimeter.

3. ANALYTICAL SOLUTION

3.1. Finite-Integral transform method

The differential problem defined by equation (7), subject to the boundary conditions specified for all the thermal versions that have been considered, is linear and its solution can be tackled by the finite-integral transform technique.

TABLE I
Coefficients d_{Ni} for the T and the H1 problem.

Version	d_{1x}	d_{2x}	d_{3x}	d_{4x}	d_{1y}	d_{2y}	d_{3y}	d_{4y}
1L	0	1	0	1	1 (or 0)	0 (or 1)	0 (or 1)	1 (or 0)
1S	1 (or 0)	0 (or 1)	0 (or 1)	1 (or 0)	0	1	0	1
2L	0	1	0	1	1	0	1	0
2S	1	0	1	0	0	1	0	1
2C	1 (or 0)	0 (or 1)	0 (or 1)	1 (or 0)	1 (or 0)	0 (or 1)	0 (or 1)	1 (or 0)
3L	1 (or 0)	0 (or 1)	0 (or 1)	1 (or 0)	1	0	1	0
3S	1	0	1	0	1 (or 0)	0 (or 1)	0 (or 1)	1 (or 0)
4	1	0	1	0	1	0	1	0

In order to solve the temperature problem given by equation (7) with the eigenfunction expansion technique, the appropriate eigenvalue problem is taken as:

$$\begin{cases} \dfrac{\mathrm{d}\Phi_{i,n}}{\mathrm{d}i} + \lambda_{i,n}^2 \Phi_{i,n} = 0 \\ d_{1i}\Phi_{i,n}(0) + d_{2i}\left.\dfrac{\mathrm{d}\Phi_{i,n}}{\mathrm{d}i}\right|_0 = 0 \\ d_{3i}\Phi_{i,n}(\delta) + d_{4i}\left.\dfrac{\mathrm{d}\Phi_{i,n}}{\mathrm{d}i}\right|_\delta = 0 \end{cases} \tag{10}$$

with $\delta = \begin{cases} 1 \text{ if } i = x \\ \beta \text{ if } i = y \end{cases}$

where i is equal to x or y and n is the order of the general eingevalue (λ) and of the general eigenfunction (Φ) that fulfils the Sturm–Liouville problem defined by equation (10). The coefficients d_{Ni} depend on the specified combination of heated and adiabatic walls, imposed by the boundary conditions. With reference to the eight thermal versions considered, the values assumed by the constants d_{Ni} are shown in *table I* for the T and the H1 problems (it is easy to demonstrate that for the T and the H1 problems the coefficients d_{Ni} assume the same values) and in *table II* for the H2 problem.

In heat conduction problems the infinite series of eigenvalues $\lambda_{i,n}$ and the related eigenfunctions $\Phi_{i,n}$ that resolve equation (10) are used frequently and their expressions can be found in all the textbooks on heat conduction [13, 14].

For the sake of completeness, *table III* shows the eigenvalues and the eigenfunctions generated by equation (10) for all the thermal versions considered of the T and the H1 problems. In the same way one can find the eigenvalues and the eigenfunctions for the H2 problem in *table IV*.

TABLE II
Coefficients d_{Ni} for the H2 problem.

Version	d_{1x}	d_{2x}	d_{3x}	d_{4x}	d_{1y}	d_{2y}	d_{3y}	d_{4y}
for any version	0	1	0	1	0	1	0	1

3.2. The T problem

If one knows the eigenvalues and the eigenfunctions defined by the appropriate Sturm–Liouville problem (equation (10)) for any T thermal version considered, then by applying the method of separation of variables, it is easy to demonstrate that equation (7) reads:

$$\lambda_{x,n}^2 + \lambda_{y,m}^2 = c^* \, Nu_{n,m} \tag{11}$$

hence the T problem admits a discrete spectrum of Nusselt numbers that fulfil the differential problem defined by equation (7).

It is interesting to notice that, for any Nusselt number $Nu_{n,m}$ defined by equation (11), one matched temperature distribution exists which can be written as follows:

$$T_{n,m} = T(x,y,Nu_{n,m}) = A\,\Phi_{x,n}\,\Phi_{y,m} \tag{12}$$

where A is an arbitrary constant.

It is simple to demonstrate that only in correspondence of the first eigenvalues does the temperature distribution not change its sign in the cross section of the duct and hence it has an actual physical meaning.

Accordingly, it is possible to write that, for the T problem, the temperature field turns out to be:

$$T(x,y) = A\,\Phi_{x,1}\,\Phi_{y,1} \tag{13}$$

TABLE III Eigenvalues and eigenfunctions for the T and the H1 problems.				
Version	$\lambda^2_{x,n}$	$\lambda^2_{y,m}$	$\Phi_{x,n}$	$\Phi_{y,m}$
1L	$n^2\pi^2$	$(2m-1)^2\dfrac{\pi^2}{4\beta^2}$	$\cos(\lambda_{x,n}x)$	$\sin(\lambda_{y,m}y)$
1S	$(2n-1)^2\dfrac{\pi^2}{4}$	$\dfrac{m^2\pi^2}{\beta^2}$	$\sin(\lambda_{x,n}x)$	$\cos(\lambda_{y,m}y)$
2L	$n^2\pi^2$	$\dfrac{m^2\pi^2}{\beta^2}$	$\cos(\lambda_{x,n}x)$	$\sin(\lambda_{y,m}y)$
2S	$n^2\pi^2$	$\dfrac{m^2\pi^2}{\beta^2}$	$\sin(\lambda_{x,n}x)$	$\cos(\lambda_{y,m}y)$
2C	$(2n-1)^2\dfrac{\pi^2}{4}$	$(2m-1)^2\dfrac{\pi^2}{4\beta^2}$	$\sin(\lambda_{x,n}x)$	$\sin(\lambda_{y,m}y)$
3L	$(2n-1)^2\dfrac{\pi^2}{4}$	$\dfrac{m^2\pi^2}{\beta^2}$	$\sin(\lambda_{x,n}x)$	$\sin(\lambda_{y,m}y)$
3S	$n^2\pi^2$	$(2m-1)^2\dfrac{\pi^2}{4\beta^2}$	$\sin(\lambda_{x,n}x)$	$\sin(\lambda_{y,m}y)$
4	$n^2\pi^2$	$\dfrac{m^2\pi^2}{\beta^2}$	$\sin(\lambda_{x,n}x)$	$\sin(\lambda_{y,m}y)$

TABLE IV Eigenvalues and eigenfunctions for the H2 problem.				
Version	$\lambda^2_{x,n}$	$\lambda^2_{y,m}$	$\Phi_{x,n}$	$\Phi_{y,m}$
for any version	$n^2\pi^2$	$\dfrac{m^2\pi^2}{\beta^2}$	$\cos(\lambda_{x,n}x)$	$\cos(\lambda_{y,m}y)$

Hence, the Nusselt numbers can be calculated as follows:

$$Nu_T = \frac{\lambda^2_{x,1} + \lambda^2_{y,1}}{c^*} \tag{14}$$

Table V shows the expressions assumed by c^* and the Nusselt numbers for all the different thermal versions of the T problem considered here.

3.3. The H1 problem

By using the appropriate eigenfunctions defined for any different thermal version of the H1 problem (*table III*), the unknown temperature field is sought by resorting to a double series:

$$T(x,y,z) = T_w(z) + \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} B_{n,m}\,\Phi_{x,n}\,\Phi_{y,m} \tag{15}$$

In fact, the H1 boundary condition states that the wall temperature $T_w(z)$ is uniform on the heated length of a rectangular perimeter and that it increases linearly with the longitudinal coordinate z [15]. The temperature distribution is defined with the exception of an additive constant if one determines the constants $B_{n,m}$. In order to obtain the solution to equation (7), the first step consists of multiplying every term of the energy equation by $\Phi_{x,n}\,\Phi_{y,m}$ and then integrating over x, between 0 and 1, and over y, between 0 and β. The integrals appearing in this procedure can be easily and patiently solved by the classic methods inasmuch as the eigenfunctions quoted in *table III* are very simple trigonometric functions. After some algebra it is possible to obtain the unknown coefficients for all the thermal versions of the H1 problem examined. The coefficients $B_{n,m}$ of equation (15) are given in *table VI*.

Using equations (8)–(9) one can calculate the bulk temperature and the Nusselt numbers for any version of the H1 problem (*table VII*).

3.4. The H2 problem

The eigenfunctions quoted in *table IV* do not depend on the thermal version of the H2 problem considered. The unknown temperature is sought by resorting to a double series:

$$T(x,y,z) = C_{0,0} + \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} C_{n,m}\cos(n\pi x)\cos\left(\frac{m\pi y}{\beta}\right) \tag{16}$$

As is well known, the temperature distribution in the H2 boundary conditions is defined apart from

TABLE V
The c^* coefficients of equation (7) and Nusselt numbers for the T thermal problem.

Version	c^*	Nu_T	Version	c^*	Nu_T
1L	$\frac{1+\beta}{2\beta^2}$	$\frac{\pi^2(4\beta^2+1)}{2(1+\beta)}$	2C	$\frac{1}{2}\left(\frac{1+\beta}{\beta}\right)^2$	$\frac{\pi^2(\beta^2+1)}{2(1+\beta)^2}$
1S	$\frac{1+\beta}{2\beta}$	$\frac{\pi^2(\beta^2+4)}{2\beta(1+\beta)}$	3L	$\frac{(1+\beta)(2+\beta)}{2\beta^2}$	$\frac{\pi^2(\beta^2+4)}{2(1+\beta)(2+\beta)}$
2L	$\frac{1+\beta}{\beta^2}$	$\frac{\pi^2(\beta^2+1)}{(1+\beta)}$	3S	$\frac{(1+\beta)(1+2\beta)}{2\beta^2}$	$\frac{\pi^2(4\beta^2+1)}{2(1+\beta)(1+2\beta)}$
2S	$\frac{1+\beta}{\beta}$	$\frac{\pi^2(\beta^2+1)}{\beta(1+\beta)}$	4	$\left(\frac{1+\beta}{\beta}\right)^2$	$\frac{\pi^2(\beta^2+1)}{(1+\beta)^2}$

TABLE VI
The coefficients $B_{n,m}$ of equation (15).

Version	$B_{n,m}$	Version	$B_{n,m}$
1L	$-\frac{16\beta}{\pi^3}\frac{1}{m^3}$	2C	$-\frac{64\beta}{\pi^4}\frac{1}{nm(m^2+\beta^2 n^2)}$
1S	$-\frac{16}{\pi^3\beta}\frac{1}{n^3}$	3L	$-\frac{64\beta}{\pi^4}\frac{1}{nm(4m^2+\beta^2 n^2)}$
2L	$-\frac{4\beta}{\pi^3}\frac{1}{m^3}$	3S	$-\frac{64\beta}{\pi^4}\frac{1}{nm(4n^2\beta^2+m^2)}$
2S	$-\frac{4}{\pi^3\beta}\frac{1}{n^3}$	4	$-\frac{16\beta}{\pi^4}\frac{1}{nm(n^2\beta^2+m^2)}$
For all the versions $B_{n,m}=0$ if n and m are not odd.			

an additive constant ($C_{0,0}$), because no temperature value is given either on the boundary or in the cross section. On the contrary, the Nusselt number, defined by equation (9), depends on a temperature difference and hence is univocally defined.

In order to obtain the solution to equation (7), one can use the same method followed in the resolution of the H1 problem, bearing in mind that, in the integration by parts one must allow for the boundary conditions that are not homogeneous (see Spiga and Morini [12]):

$$\left.\frac{\partial T}{\partial x}\right|_{x=0} = \gamma_1\,\frac{(1+\beta)}{2\beta}, \quad \left.\frac{\partial T}{\partial x}\right|_{x=1} = \gamma_2\,\frac{(1+\beta)}{2\beta}$$
$$\left.\frac{\partial T}{\partial y}\right|_{y=0} = \gamma_3\,\frac{(1+\beta)}{2\beta}, \quad \left.\frac{\partial T}{\partial y}\right|_{y=\beta} = \gamma_4\,\frac{(1+\beta)}{2\beta} \tag{17}$$

where the coefficients γ_N depend on the thermal version of the H2 problem considered and assume the value 0 for adiabatic walls, +1 or −1 for heated walls (the sign depending on the direction of the temperature gradient). After some algebra it is possible to obtain the unknown temperature distribution as a function of the γ_N coefficients (Gao and Hartnett [11]):

$$T(x,y) = \frac{\gamma_2\,(1+\beta)}{4\beta}\left(x^2-\frac{1}{3}\right) - \frac{\gamma_1\,(1+\beta)}{2\beta}\left(\frac{1}{3}-x+\frac{x^2}{2}\right) + \frac{\gamma_4\,(1+\beta)}{4}\left(\frac{y^2}{\beta^2}-\frac{1}{3}\right) - \frac{\gamma_3\,(1+\beta)}{2}\left(\frac{1}{3}-\frac{y}{\beta}+\frac{y^2}{2\beta^2}\right) \tag{18}$$

It can be observed that, in versions 1L, 2L, 1S, and 2S, the dimensionless temperature depends on only one co-ordinate y (1L, 2L) or x (1S, 2S). For version 4 the temperature profile is symmetric with respect to the centre of the cross section, while in the other versions the 2-D temperature profile is not symmetric, due to the boundary conditions. By applying its definition (equation (9)), the Nusselt number reads as [12]:

$$Nu_{H2} = \frac{6\,L_h^*}{(1+\beta)\left[(\gamma_1+\gamma_2)^2+(\gamma_3+\gamma_4)^2-\gamma_1\gamma_2-\gamma_3\gamma_4\right]} \tag{19}$$

4. RESULTS AND DISCUSSION

The previous results have been worked out on a PC equipped with a 80486 processor. For the T and H2 problems the temperature distributions and hence the Nusselt numbers can be calculated easily by making a simple spreadsheet. For the H1 problem by contrast, the temperature distribution is sought by resorting to a uniform convergent double series (equation (15)). The fast convergence of the double series is assured by the third power of n and m in the denominator of the $B_{n,m}$ coefficients quoted in *table VI* and makes the present technique quite effective and inexpensive in terms of computer time. It is interesting to note that the T temperature distribution can be regarded as the first

TABLE VII Nusselt numbers for the H1 problem.			
Version	Nu_{H1}	Version	Nu_{H1}
1L	$\frac{6}{1+\beta}$	2C	$\frac{\pi^6}{128\,(1+\beta)^2 \sum_{n=1,odd}^{\infty}\sum_{m=1,odd}^{\infty}\frac{1}{n^2m^2\,(m^2+\beta^2n^2)}}$
1S	$\frac{6\,\beta}{1+\beta}$	3L	$\frac{\pi^6}{128\,(1+\beta)\,(2+\beta) \sum_{n=1,odd}^{\infty}\sum_{m=1,odd}^{\infty}\frac{1}{n^2m^2\,(4\,m^2+\beta^2n^2)}}$
2L	$\frac{12}{1+\beta}$	3S	$\frac{\pi^6}{128\,(1+\beta)\,(2\,\beta+1) \sum_{n=1,odd}^{\infty}\sum_{m=1,odd}^{\infty}\frac{1}{n^2m^2\,(m^2+4\,\beta^2n^2)}}$
2S	$\frac{12\,\beta}{1+\beta}$	4	$\frac{\pi^6}{64\,(1+\beta)^2 \sum_{n=1,odd}^{\infty}\sum_{m=1,odd}^{\infty}\frac{1}{n^2m^2\,(m^2+\beta^2n^2)}}$

term of the double series used to formulate the H1 temperature distribution:

$$T(x,y) = B_{1,1}\,\Phi_{x,1}\,\Phi_{y,1} \tag{20}$$

where the arbitrary constant A of equation (13) is set equal to the first coefficient $B_{1,1}$ determined for the H1 problem (*table VI*). In other words, this means that the temperature distribution for the H1 boundary condition is obtained by starting from the T distribution with the addition of 'remedial' terms.

In *figure 1* the dimensionless temperature distributions for a square duct with the four heated walls for the T, H1 and H2 boundary conditions are quoted.

The temperature distribution around the duct periphery is uniform for the T and H1 problems, whereas for the H2 boundary condition the wall temperature is strongly variable along the duct perimeter. In this latter case the maximum wall temperature occurs at the corners of the cross sectional duct and the minimum wall temperature occurs at the midpoint of the long side (*figures 1* and *2*) of the rectangular duct.

In *figure 2* we show the dimensionless temperature distributions for a rectangular duct with aspect ratio $\beta = 0.25$ and four heated walls (version 4). The aspect ratio effect on the temperature distributions for the three boundary conditions considered is evident; for the H2 boundary condition one can observe that, for small aspect ratio, the minimum wall temperature approaches the minimum fluid temperature which occurs at the centre of the rectangular duct. For the T and H1 problems, it turns out that in the rectangular duct with small aspect ratio the temperature distribution experiences a strong decrease near the heated walls and an extended central core with low temperature. *Figure 2* shows how this effect is more evident for the H1 problem that for the T boundary condition.

Figure 3 shows the dimensionless temperature profiles along the diagonal of a square duct for T and H1 boundary conditions (version 4). It can be seen that the dimensionless temperature field for the T problem presents an inflection point near the corners of the duct walls. For the H1 boundary condition the wall temperature is continuously 'running away' [7] from the bulk temperature so that an inflection in the temperature profile does not develop. For the same difference in both the wall temperature and the bulk temperature, the fluid temperature gradient at the wall is smaller for the T boundary condition because of the inflection. Thus, from equation (9) the Nusselt numbers are lower for the T problem and higher for the H1 problem, as can be observed in *figure 4*.

Figure 5 shows the modulus of the local dimensionless temperature gradient ($|\nabla T|$) on the cross section ($\beta = 0.25$, version 4) for the three boundary conditions considered; one can see the different role played by the duct's corners on the 2D-field of the heat flux for the T, H1 and H2 boundary conditions.

For the H2 problem, since the corner temperature is higher, the peripheral average wall temperature is greater; this leads to a reduction of the H2 Nusselt numbers with respect to the H1 and T boundary conditions.

In *figure 4* the Nusselt numbers are reported for the T, H1 and H2 problems for slug flow and for fully developed laminar profile (version 4); it is interesting to notice that the trend of the Nusselt number as a function of the aspect ratio is the same for slug flow and

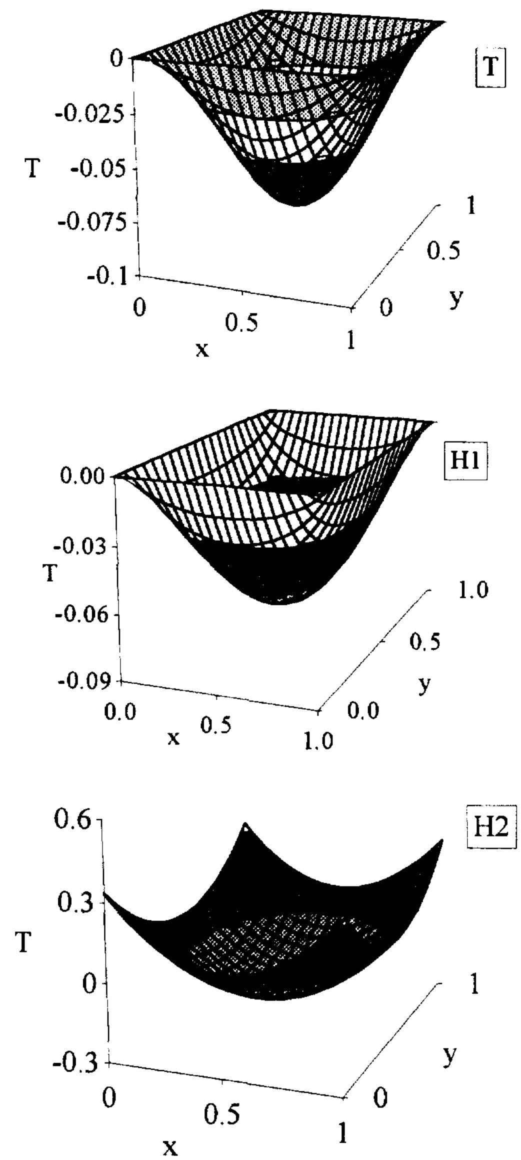

Figure 1. Dimensionless temperature distributions in a square duct with four sides heated (version 4) for the T, H1 and H2 problems.

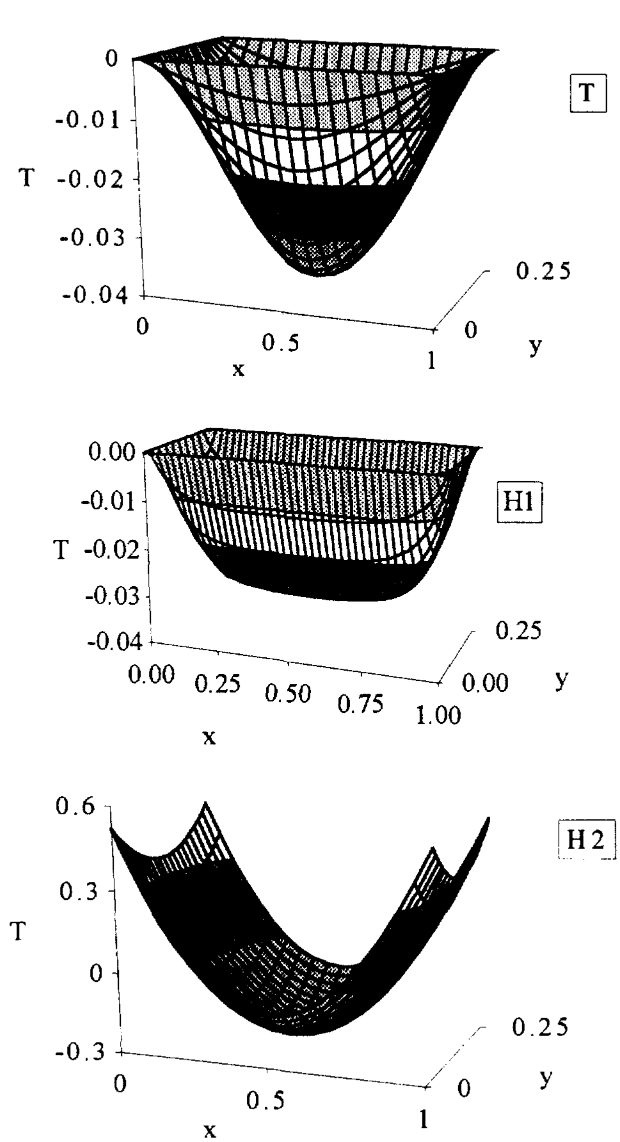

Figure 2. Dimensionless temperature distributions in a rectangular duct ($\beta = 0.25$) with four sides heated (version 4) for the T, H1 and H2 problems.

for laminar fully developed velocity profile. To stress the fundamental role played by the velocity distribution on the thermal behaviour it is pointed out that, for the slug flow, the H2 Nusselt numbers are nearly twice the H2 Nusselt numbers for the fully developed profile. The difference in the two values assumed by the Nusselt numbers when β varies, for the H1 and H2 problems, increases when the aspect ratio tends to zero because for small aspect ratio the corner effect on the temperature distribution becomes very strong. The Nusselt numbers for the T and H1 problems are quoted in *tables VIII* and *IX* respectively for the 1L, 2L, 1S, 2S versions; in *tables X* and *XI* the T and H1 Nusselt numbers are compared with the numerical results given by Hartnett and Kostic [10] for the 4, 3L, 3S and 2C versions; it is interesting to notice how the numerical results of Hartnett and Kostic are in good agreement with the analytical solutions presented in this paper. The Nusselt numbers for the H2 boundary conditions are quoted in *table XII* and coincide with the values given by Gao and Hartnett [11] and by Spiga and Morini [12].

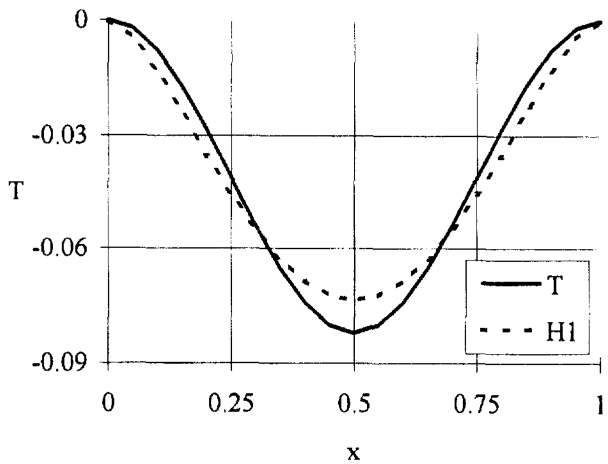

Figure 3. Dimensionless temperature profiles along the diagonal of a square duct with four sides heated (version 4) for the T and H1 problems.

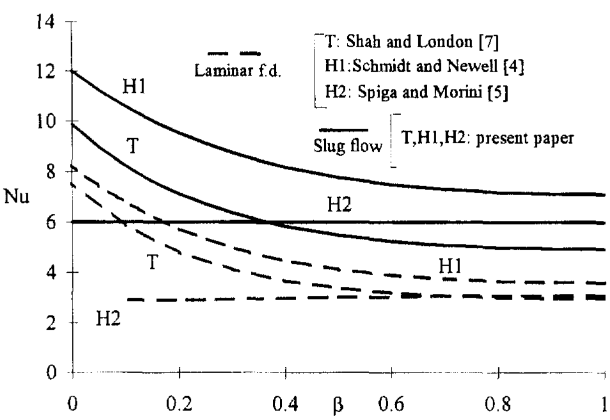

Figure 4. Nusselt numbers for the T, H1 and H2 problems for slug and fully developed flow in a square duct (version 4) as a function of the duct aspect ratio.

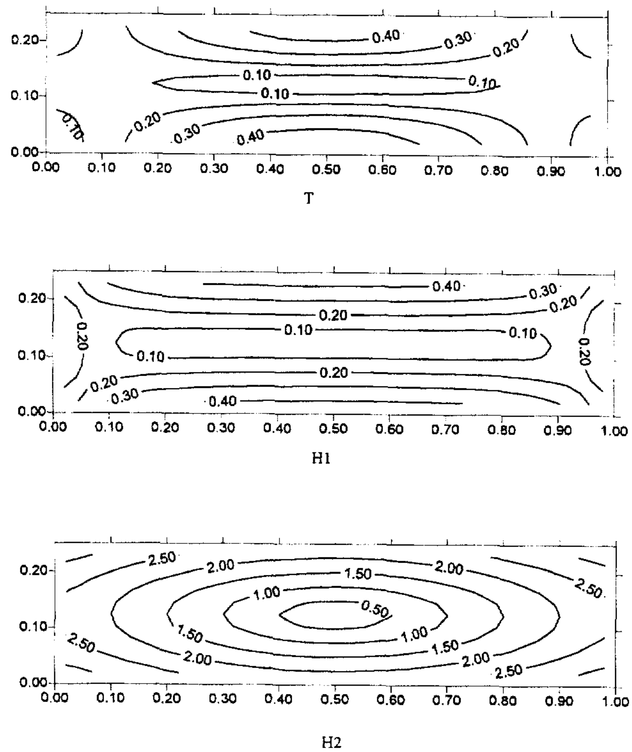

Figure 5. The 2D-distribution of the local dimensionless temperature gradient ($|\nabla T|$) on the cross section of a rectangular duct ($\beta = 0.25$) with four sides heated for the T, H1 and H2 problems.

If the temperature distributions and the Nusselt numbers for the 1L,2L, 1S and 2L versions of the H1 and H2 problems are compared, it will be found that for such versions it is not possible to distinguish the H1 and H2 boundary conditions as is the case for a slab or a circular duct. In other words, for slug flow in a rectangular duct the boundary conditions H1 and H2 produce the same results only if no adjacent wall of duct is heated. This fact is not true for a rectangular duct where a laminar fully developed velocity profile occurs, as it does in the research carried out by Spiga and Morini [5]. It is of interest to note that, using the expressions quoted in *table V* for the T Nusselt numbers, the Nusselt number as a function of the aspect ratio achieves a minimum for versions 1L, 2L 2C, 3S and 4. For the 1L version the Nusselt number is minimum when β is equal to $(5/4)^{0.5} - 1$ ($\beta = 0.118$); for version 2L the aspect ratio $\beta = 2^{0.5} - 1$ (0.4142) minimises the Nusselt number, whereas for versions 2C and 4 the Nusselt number is minimum for square ducts. Finally, for version 3S, Nu reaches its minimum value when β is equal to $(5/18)^{0.5} - 1/6$ ($\beta = 0.36$). For the H1 problem the Nusselt number achieves a minimum only for the version 3S ($\beta = 0.34$), while a value of the aspect ratio that minimises the Nusselt number for any version of the H2 problem does not exist.

5. CONCLUDING REMARKS

The paper contains an analytical study of heat transfer to slug flow in the thermal fully developed region of rectangular ducts, for the T, H1 and H2 boundary conditions. The 2-D temperature distribution has been analytically determined for all the different combinations of heated and adiabatic walls of practical interest. The Nusselt numbers are accurately predicted and compared with the results obtained numerically by several authors.

TABLE VIII
Nusselt numbers for the T problem.

	Nu_T			
β	1L	1S	2L	2S
1	12.337	12.337	9.870	9.870
0.9	11.012	13.881	9.402	10.447
0.8	9.760	15.901	8.992	11.240
0.7	8.592	18.620	8.650	12.358
2/3	8.225	19.739	8.554	12.830
0.6	7.526	22.412	8.389	13.982
0.5	6.580	27.964	8.225	16.449
0.4	5.781	36.659	8.178	20.444
1/3	5.346	45.647	8.225	24.674
0.3	5.163	51.752	8.275	27.584
1/4	4.935	64.152	8.389	33.557
0.2	4.770	83.069	8.554	42.768
1/6	4.700	102.221	8.695	52.168
1/8	4.661	140.916	8.910	71.280
0.1	4.666	179.896	9.062	90.621
1/16	4.717	297.540	9.325	149.205
0.05	4.747	376.220	9.423	188.462
0	4.935	∞	9.870	∞

TABLE IX
Nusselt numbers for the H1 problem.

	Nu_{H1}			
β	1L	1S	2L	2S
1	3	3	6	6
0.9	3.158	2.842	6.316	5.684
0.8	3.333	2.667	6.667	5.333
0.7	3.529	2.471	7.059	4.941
2/3	3.6	2.4	7.2	4.8
0.6	3.75	2.25	7.5	4.5
0.5	4	2	8	4
0.4	4.286	1.714	8.571	3.429
1/3	4.5	1.5	9	3
0.3	4.615	1.385	9.231	2.769
1/4	4.8	1.2	9.6	2.4
0.2	5	1	10	2
1/6	5.143	0.857	10.286	1.714
1/8	5.333	0.667	10.667	1.333
0.1	5.455	0.545	10.909	1.091
1/16	5.647	0.353	11.294	0.706
0.05	5.714	0.286	11.429	0.571
0	6	0	12	0

TABLE X
Comparison of the Nusselt numbers for the T problem with the numerical results in the literature.

	Nu_T							
	4		3L		3S		2C	
β		Hartnett and Kostic [10]		Hartnett and Kostic [10]		Hartnett and Kostic [10]		Hartnett and Kostic [10]
1	4.935	4.94	4.112	4.11	4.112	4.11	2.467	2.47
0.9	4.948	–	4.308	–	3.933	–	2.474	–
0.8	4.996	–	4.543	–	3.754	–	2.498	–
0.7	5.088	–	4.827	–	3.580	–	2.544	–
0.6	5.243	–	5.172	–	3.421	–	2.622	–
0.5	5.483	5.48	5.593	5.62	3.290	3.29	2.742	2.74
0.4	5.841	–	6.110	–	3.212	–	2.921	–
0.3	6.366	–	6.750	–	3.227	–	3.183	–
1/4	6.711	6.74	7.128	7.19	3.290	3.29	3.356	3.37
0.2	7.128	–	7.552	–	3.407	–	3.564	–
1/8	7.920	7.99	8.289	–	3.729	3.74	3.960	4.00
0.1	8.238	–	8.566	–	3.888	–	4.119	–
1/16	8.777	–	9.016	–	4.193	4.23	4.388	–
0	9.870	9.87	9.870	9.87	4.935	4.94	4.935	4.94

TABLE XI
Comparison of the Nusselt numbers for the H1 problem with the numerical results in the literature.

	Nu_T							
	4		3L		3S		2C	
β		Hartnett and Kostic [10]		Hartnett and Kostic [10]		Hartnett and Kostic [10]		Hartnett and Kostic [10]
1	7.114	7.11	5.831	5.82	5.831	5.83	3.557	3.56
2/3	7.356	7.36	6.836	6.84	5.066	5.07	3.678	3.68
0.5	7.774	7.77	7.597	7.60	4.742	4.74	3.887	3.88
0.4	8.184	8.18	8.173	8.17	4.621	4.62	4.092	4.09
1/3	8.545	8.55	8.620	8.63	4.597	4.60	4.272	4.27
1/4	9.116	9.12	9.263	9.27	4.664	4.66	4.558	4.56
0.2	9.535	9.54	9.702	9.70	4.774	4.77	4.768	4.77
1/6	9.851	9.86	10.021	10.03	4.883	4.88	4.926	4.93
1/8	10.292	10.30	10.451	10.46	5.065	5.06	5.146	5.15
0.1	10.584	10.58	10.728	10.728	5.201	5.20	5.292	5.29
1/12	11.066	11.07	11.172	11.18	5.449	5.46	5.533	5.54
0.05	11.239	11.24	11.328	11.34	5.544	5.54	5.619	5.62
0	12	12	12	12	6	6	6	6

TABLE XII
Nusselt numbers for the H2 problem.

	Nu_{H2}							
β	4	3L	3S	2C	2L	2S	1L	1S
1	6	4.5	4.5	3	6	6	3	3
0.9	6	4.579	4.421	3	6.316	5.684	3.158	2.842
0.8	6	4.667	4.333	3	6.667	5.333	3.333	2.667
0.7	6	4.765	4.235	3	7.059	4.941	3.529	2.471
2/3	6	4.8	4.2	3	7.2	4.8	3.6	2.4
0.6	6	4.875	4.125	3	7.5	4.5	3.750	2.250
0.5	6	5	4	3	8	4	4	2
0.4	6	5.143	3.857	3	8.571	3.429	4.286	1.714
1/3	6	5.250	3.750	3	9	3	4.5	1.5
0.3	6	5.308	3.692	3	9.231	2.769	4.615	1.385
1/4	6	5.4	3.6	3	9.6	2.4	4.8	1.2
0.2	6	5.5	3.5	3	10	2	5	1
1/6	6	5.571	3.429	3	10.286	1.714	5.143	0.857
1/8	6	5.667	3.333	3	10.667	1.333	5.333	0.667
0.1	6	5.727	3.273	3	10.909	1.091	5.455	0.545
1/16	6	5.824	3.176	3	11.294	0.706	5.647	0.353
0.05	6	5.857	3.143	3	11.429	0.571	5.714	0.286
0	6	6	3	3	12	0	6	0

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